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STOCHASTIC CONTROL IN A DIFFERENTIAL GAME

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A stochastic differential game with a fixed termination time is considered within the framework of the formalization in [1]. The existence of saddle points of the stochastic differential game in classes of position strategies is proved. The convergence of the stochastic differential game's value to that of an ordinary position differential game as the noise intensity decreases to zero is established. The possibility of using a stochastic process as guide in the control procedures is shown for a position differential game. This paper follows [2, 3].

1. Let the motion of a conflict-controlled system be described by the stochastic differential equation

$$dx[t] = f(t, x, u, v)dt + \varkappa dz[t], \quad x[t_0] = x_0 \tag{1.1}$$
$$u \in P \subset R^p, \; v \in Q \subset R^q, \; f: [t_0, \vartheta] \times R^n \times P \times Q \to R^n$$

Here x is the phase vector from R^n, u and v are the controls, and P and Q are closed bounded sets. Function f is continuous, uniformly bounded and satisfies a uniform Lipschitz conditions in x from R^n, i. e.,

$$\|f(t, x^{(1)}, u, v) - f(t, x^{(2)}, u, v)\| \leqslant \lambda \|x^{(1)} - x^{(2)}\|$$
$$\forall (t, x^{(i)}, u, v) \in [t_0, \vartheta] \times R^n \times P \times Q \quad (i = 1, 2)$$

The symbol $\|\xi\|$ denotes the Euclidean norm of vector ξ, ϑ is some finite time instant, $z[t]$ is a vector-valued Wiener process whose components $z^{(i)}[t]$ $(i = 1, 2, \ldots, n)$ are independent standard Wiener processes (see [4]), and $\varkappa$ is a constant.

Let a continuous bounded function $\sigma: R^n \to R$ be specified, having continuous derivatives of up to second order; σ''_{xx} satisfies the Hölder condition

$$\|\sigma''_{xx}(x^{(1)}) - \sigma''_{xx}(x^{(2)})\| < K \|x^{(1)} - x^{(2)}\|^{\psi}, \; 0 < \psi < 1, \quad K = \text{const}$$

The mathematical expectation of the random value $\sigma(x[\vartheta])$ serves as the payoff in the stochastic differential game. These games of the form (1.1) were examined in [2, 3, 5-8]. In the present paper a stochastic differential game is set and investigated in accordance with the formalization of players' strategies, presented in [1]. Three basic types of position strategies are examined, corresponding to the three cases of availability of information to the players in the differential game. The existence is established of saddle points of the stochastic differential game in the class of pure position strategies of one player and of counter-strategies of the other, as well as in the class of mixed position strategies of both players. It is shown that in the limit as $\varkappa \to 0$ the stochastic differential game's value yields the position differential game's value. A procedure is proposed for solving position game control problems on the basis of the solution of an auxiliary stochastic game.

We define the class of first player's strategies U.

1°. Pure position strategies U are identified with Borel-measurable (see [9]) functions $U : [t_0, \vartheta] \times R^n \to P$. The totality of first player's pure position strategies is denoted $\{U\}_1$.

2°. The first player's mixed position strategies U are identified with the functions $U : [t_0, \vartheta] \times R^n \to P$, where $\overline{P}$ is the set of probability measures μ normed on compactum P. It is assumed that these functions are Borel-weakly-measurable. The totality of first player's mixed position strategies is denoted $\{U\}_2$.

3°. The first player's counter-strategies are identified with Borel-measurable functions $U : [t_0, \vartheta] \times R^n \times Q \to P$. The totality of them is denonted $\{U\}_3$.

The second player's strategy classes $\{V\}_1$, $\{V\}_2$, and $\{V\}_3$ are defined analogously with the letters U, μ, and P replaced by V, ν, and Q and with the indices 1, 2 and 3 permuted to 3, 2 and 1. We examine the following three sets of strategy pairs:

$$\{U\}_1 \times \{V\}_1, \quad \{U\}_2 \times \{V\}_2, \quad \{U\}_3 \times \{V\}_3$$

For any strategy pair $(U, V) \in \{U\}_k \times \{V\}_k (k = 1, 2, 3)$ we can define the corresponding weak solution $x_\omega[t] = x_\omega[t; t_0, x_0, U, V]$ $(t_0 \leqslant t \leqslant \vartheta)$ of Eq. (1.1) (see [4]), satisfying with probability one the equality

$$x_\omega[t] = x_0 + \int_{t_0}^{t} f(\tau, x_\omega[\tau], u_\omega[\tau], v_\omega[\tau])\, d\tau + \int_{t_0}^{t} \varkappa\, dz_\omega[\tau] \quad (k = 1, 3) \tag{1.2}$$

$$(u_\omega[\tau] = U(\tau, x_\omega[\tau]),\ v_\omega[\tau] = V(\tau, x_\omega[\tau], u_\omega[\tau]),\ k = 1)$$

$$(u_\omega[\tau] = U(\tau, x_\omega[\tau], v_\omega[\tau]),\ v_\omega[\tau] = V(\tau, x_\omega[\tau]),\ k = 3)$$

$$x_\omega[t] = x_0 + \int_{t_0}^{t} \int_P \int_Q f(\tau, x_\omega[\tau], u, v)\, \mu_{\omega,\tau}(du)\, \nu_{\omega,\tau}(dv)\, d\tau + \int_{t_0}^{t} \varkappa dz_\omega[\tau] \quad (k = 2)$$

$$(\mu_{\omega,\tau} = U(\tau, x_\omega[\tau]),\ \nu_{\omega,\tau} = V(\tau, x_\omega[\tau]))$$

Problem 1_k. The initial position (t_0, x_0) is specified. Find the strategy $U_k^e \in \{U\}_k$, ensuring the fulfilment of the equality

$$\max_V M[\sigma(x[\vartheta; t_0, x_0, U_k^e, V])] = \min_U \max_V M[\sigma(x[\vartheta; t_0, x_0, U, V])]$$
$$(U, V) \in \{U\}_k \times \{V\}_k \quad (k = 1, 2, 3)$$

Problem 2_k. The initial position (t_0, x_0) is specified. Find the strategy $V_k^e \in \{V\}_k$, ensuring the fulfilment of the equality

$$\min_U M[\sigma(x[\vartheta; t_0, x_0, U, V_k^e])] = \max_V \min_U M[\sigma(x[\vartheta; t_0, x_0, U, V])]$$
$$(U, V) \in \{U\}_k \times \{V\}_k \quad (k = 1, 2, 3)$$

2. Let us consider a partial differential equation with boundary condition

$$\frac{\partial\gamma}{\partial t} + \frac{1}{2}\varkappa^2\nabla^2\gamma + F_k(t, x, \nabla\gamma) = 0 \quad (k = 1, 2, 3) \tag{2.1}$$
$$\nabla^2\gamma = \sum_{i=1}^{n} \frac{\partial^2\gamma(t, x)}{\partial x_i^2}$$
$$\gamma(\vartheta, x) = \sigma(x), \qquad x \in R^n$$
$$F_k : [t_0, \vartheta] \times R^n \times R^n \to R$$
$$F_1(t, x, s) = \min_{u\in P} \max_{v\in Q} s'f(t, x, u, v)$$
$$F_2(t, x, s) = \min_{\mu\in\overline{P}} \max_{\nu\in\overline{Q}} \int_P \int_Q s'f(t, x, u, v)\,\mu(du)\,\nu(dv)$$
$$F_3(t, x, s) = \max_{v\in Q} \min_{u\in P} s'f(t, x, u, v), \quad s \in R^n$$

Here $\nabla\gamma$ is the gradient of function γ with respect to variable x; $x_i (i = 1, 2, \ldots, n)$ are the components of phase vector x; the prime denotes transposition. The functions $F_k (k = 1, 2, 3)$ are continuous and satisfy a Lipschitz condition in x and s and, therefore (see [10]), a solution of the parabolic Eq. (2.1) exists.

Let us determine the extremal strategies $U_k^e \in \{U\}_k$ $(k = 1, 2, 3)$. Let $\gamma_k : [t_0, \vartheta] \times R^n \to R$ be a solution of problem (2.1). We consider functions $U_k^e (k = 1, 2, 3)$ satisfying the conditions

$$\max_{\in Q} \nabla\gamma_1'(t, x) f(t, x, u_1^e, v) = F_1(t, x, \nabla\gamma_1(t, x)) \quad (k = 1) \tag{2.2}$$

$$\max_{\nu\in\bar{Q}} \int_P \int_Q \nabla\gamma_2'(t, x) f(t, x, u, v) \mu^e(du) \nu(dv) = \tag{2.3}$$
$$F_2(t, x, \nabla\gamma_2(t, x)) \quad (k = 2)$$
$$\nabla\gamma_3'(t, x) f(t, x, u_3^e, v) = \min_{u\in P} \nabla\gamma_3'(t, x) f(t, x, u, v) \quad (k = 3) \tag{2.4}$$
$$(u_1^e = U_1^e(t, x),\ \mu^e = U_2^e(t, x),\ u_3^e = U_3^e(t, x, v))$$
$$U_1^e : [t_0, \vartheta] \times R^n \to P,\ U_2^e : [t_0, \vartheta] \times R^n \to \bar{P},\ U_3^e : [t_0, \vartheta] \times R^n Q \to P$$

It is important that the functions $U_k^e\ (k = 1, 2, 3)$ can be chosen from conditions (2.2)-(2.4) as Borel-measurable one. Analogously, the extremal strategies $V_k^e \in \{V\}_k\ (k = 1, 2, 3)$ are defined as Borel functions whose values satisfy the conditions

$$\nabla\gamma_1'(t, x) f(t, x, u, v_1^e) = \max_{v\in Q} \nabla\gamma_1'(t, x) f(t, x, u, v) \quad (k = 1) \tag{2.5}$$
$$\min_{\mu\in\bar{P}} \int_P \int_Q \nabla\gamma_2'(t, x) f(t, x, u, v) \mu(du) \nu^e(dv) = \tag{2.6}$$
$$F_2(t, x, \nabla\gamma_2(t, x)) \quad (k = 2)$$
$$\min_{u\in P} \nabla\gamma_3'(t, x) f(t, x, u, v_3^e) = F_3(t, x, \nabla\gamma_3(t, x)) \quad (k = 3) \tag{2.7}$$
$$(V_1^e(t, x, u) = v_1^e \in Q,\ V_2^e(t, x) = \nu^e \in \bar{Q},\ V_3^e(t, x) = v_3^e \in Q)$$

The following statement is valid.

Theorem 2.1. The extremal strategies U_k^e and V_k^e from (2.2)-(2.7) solve Problem 1_k and Problem 2_k, respectively. They form a saddle point of the stochastic differential game in the strategy class $\{U\}_k \times \{V\}_k$, i.e.,

$$M[\sigma(x[\vartheta; t_0, x_0, U_k^e, V])] \leqslant M[\sigma(x[\vartheta; t_0, x_0, U_k^e, V_k^e])] \leqslant$$
$$M[\sigma(x[\vartheta; t_0, x_0, U, V_k^e]))\quad (k = 1, 2, 3)$$
$$(U, V) \in \{U\}_k \times \{V\}_k,\quad M[\sigma(x[\vartheta; t_0, x_0, U_k^e, V_k^e])] =$$
$$\gamma_k(t_0, x_0)$$

Let us present the proof of this statement for $k = 1$ (Cases $k = 2, 3$ are proved similarly). Suppose that the first player had chosen the extremal strategy U_1^e defined by condition (2.2) and the second player, an arbitrary counter-strategy $V \in \{V\}_1$. The random process $\gamma_1(t, x_\omega[t])$, constructed on realizations of the random process $x_\omega[t] = x_\omega[t; t_0, x_0, U_1^e, V]$, has with probability one the stochastic differential (see [11]).

$$d\gamma_1(t, x_\omega[t]) = \frac{\partial\gamma_1(t, x_\omega[t])}{\partial t} dt + \tag{2.8}$$
$$\frac{1}{2} \varkappa^2 \nabla^2 \gamma_1(t, x_\omega[t]) dt + \left(\frac{\partial\gamma_1(t, x_\omega[t])}{\partial x}\right)' f(t, x_\omega[t], u_\omega^e[t],$$
$$v_\omega^e[t]) dt + \left(\frac{\partial\gamma_1(t, x_\omega[t])}{\partial x}\right)' \varkappa\, dz_\omega[t]$$

$$u_\omega{}^e[t] = U_1{}^e(t, x_\omega[t]), \quad v_\omega[t] = V(t, x_\omega[t], u_\omega[t]) \tag{2.9}$$

To the right-hand side of (2.8) we add and subtract the term $F_1(t, x_\omega[t], \nabla\gamma_1(t, x_\omega[t]))\,dt$. We integrate the resulting equation from t_0 to ϑ, next we compute the mathematical expectation and, using Fubini's theorem (see [9]) and the properties of a stochastic integral ([4]), we obtain

$$\begin{aligned} &M\gamma_1(\vartheta, x[\vartheta; t_0, x_0, U_1{}^e, V]) - \gamma_1(t_0, x_0) = \\ &M\Big\{\int_{t_0}^{\vartheta} [\nabla\gamma_1'(t, x_\omega[t]) f(t, x_\omega[t], u_\omega{}^e[t], v_\omega[t]) - \\ &F_1(t, x_\omega[t], \nabla\gamma_1(t, x_\omega[t]))]\,dt \end{aligned} \tag{2.10}$$

Since the integrand in (2.10) is nonpositive when the control $u_\omega{}^e[t]$ is chosen from conditions (2.2) and (2.9), we arrive at the inequality

$$\begin{aligned} &M[\gamma_1(\vartheta, x[\vartheta; t_0, x_0, U_1{}^e, V]) = \\ &\quad M[\sigma(x[\vartheta; t_0, x_0, U_1{}^e, V])] \leqslant \gamma_1(t_0, x_0) \end{aligned} \tag{2.11}$$

Now suppose that the first player chooses an arbitrary strategy $U \in \{U\}_1$ and the second player chooses the extremal strategy $V_1{}^e$ from (2.5). By analogous arguments we obtain the inequality

$$\begin{aligned} &M[\gamma_1(\vartheta, x[\vartheta; t_0, x_0, U, V_1{}^e])] = \\ &\quad M[\sigma(x[\vartheta; t_0, x_0, U, V_1{}^e])] \geqslant \gamma_1(t_0, x_0) \end{aligned} \tag{2.12}$$

The assertion of Theorem 2.1 for $k = 1$ follows from (2.11) and (2.12).

Theorem 2.1 establishes, in particular, that for a specified initial position (t_0, x_0) the stochastic differential game's value coincides with the value of function $\gamma_k(t_0, x_0)$ in each of the cases k = 1, 2, 3. Up to this point the parameter $\varkappa > 0$ of the stochastic differential game has been assumed fixed and, therefore, it was omitted in the notation for function γ_k. We now use the notation $\gamma_{k,\varkappa}$, reflecting the dependence of function γ_k on parameter $\varkappa$, and we consider the variation of the stochastic differential game's value $\gamma_{k,\varkappa}(t_0, x_0)$ for $\varkappa \to 0$.

3. Let the motion of a conflict-controlled system be described by the ordinary differential equation

$$x^\cdot = f(t, x, u, v), \quad x[t_0] = x_0, \quad t_0 \leqslant t \leqslant \vartheta \tag{3.1}$$

The game's outcome is characterized by the value $\sigma(x[\vartheta])$. Here f and σ are the functions defined in Sect. 1. A formalization of position differential games has been constructed in [1] for system (3.1). Within the framework of this formalization it was proved that a differential game defined on any of the three strategy classes $\{U\}_k \times \{V\}_k (k = 1, 2, 3)$ has the value $c_k(t_0, x_0)$. Our purpose here is to show that in the limit as $\varkappa \to 0$ the stochastic differential game's value $\gamma_{k,\varkappa}(t_0, x_0)$ yields the corresponding value $c_k(t_0, x_0)$ of the position differential game.

At first we prove the inequality

$$\varliminf_{\varkappa\to 0} \gamma_{k,\varkappa}(t_0, x_0) \geqslant c_k(t_0, x_0) \tag{3.2}$$

with $k = 1$. We define a function $V^* : [t_0, \vartheta] \times R^n \times R^n \times P \to Q$ associating a point $v^* \in Q$ satisfying the condition

$$(x-w)' f(t, x, u, v^*) = \max_{v\in Q} (x-w)' f(t, x, u, v) \tag{3.3}$$

with the point (t, x, w, u). We define as well a function $U_* : [t_0, \vartheta] \times R^n \times R^n \to P$ whose value $U_*(t, x, w) = u_* \in P$ satisfies the condition

$$\max_{v\in Q} (x-w)' f(t, x, u_*, v) = F_1(t, x, x-w) \tag{3.4}$$

It can be shown that functions V^* and U_* can be chose Borel-measurable and we subsequently assume them to be Borel functions. As before we assume that $U_1^e \in \{U\}_1$ is the extremal strategy defined by (2.2). Note that this strategy depends upon parameter $\varkappa$. Finally, let $V_1^\circ \in \{V\}_1$ be the second player's optimal counter-strategy in the position game defined in the class $\{U\}_1 \times \{V\}_1$ for system (3.1). According to [1] the inequality $\sigma(x[\vartheta; t_0, x_0, V_1^\circ]) \geqslant c_1(t_0, x_0)$ is valid for any motion $x[t; t_0, x_0, V_1^\circ]$ of system (3.1), generated by counter-strategy V_1^0.

In space $R^n \times R^n$ we consider the probabilistic process $(x_\omega[t], w_\omega[t])(t_0 \leqslant t \leqslant \vartheta, \omega \in \Omega_\vartheta)$ defined as follows. Suppose that a partitioning Δ of the interval $[t_0, \vartheta]$ by points $t_0 = \tau_0 < \tau_1 < \ldots < \tau_N = \vartheta$ has been selected. We assume that the probabilistic process $(x_\omega[t], w_\omega[t])$ $(t_0 \leqslant t \leqslant \tau_i, \omega \in \Omega_{\tau_i})$ has been defined. We select a certain realization $(x_*[t], w_*[t]) = (x_{\omega_*}[t], w_{\omega_*}[t])$ $(t_0 \leqslant t \leqslant \tau_i)$ of it and we define at first the probabilistic process $(x_\omega[t], w_\omega[t])(t_0 \leqslant t \leqslant \tau_{i+1}, \omega \in \Omega_*)$ under the condition that the selected realization $(x_*[t], w_*[t])(t_0 \leqslant t \leqslant \tau_i)$ has been fixed. The right endpoint $(x_*[\tau_i], w_*[\tau_i])$ of this realization is denoted (x_*, w_*). We consider the motion $x^*[t](\tau_i \leqslant t \leqslant \tau_{i+1})$ defined by the equality

$$x^*[t] = x_* + \int_{\tau_i}^{t} f(\tau, x^*[\tau], u[\tau], v[\tau])\, d\tau \tag{3.5}$$

$$(u[\tau] = u[\tau_i] = U_*(\tau_i, x_*, w_*),\ v[\tau] = v[\tau_i] = V_1^0(\tau_i, x_*, u[\tau_i]))$$

The equality $x_\omega[t] = x^*[t]$ for $\tau_i \leqslant t \leqslant \tau_{i+1}$ is fulfilled with probability one in the auxiliary probabilistic process.

Let the function $V^{**} : [t_0, \vartheta] \times R^n \times P \to Q$ be defined by the equality $V^{**}(t, w, u) = V^*(t, x^*[t], w, u)$, where V^* is the function chosen from condition (3.3). Clearly V^{**} is a counter-strategy from class $\{V\}_1$. Therefore, on the interval $[\tau_i, \tau_{i+1}]$ we can define a weak solution $w_\omega[t] = w_\omega[t; \tau_i, w_*, U_1^e, V^{**}]$ of the stochastic system

$$dw[t] = f(t, w, u, v)\, dt + \varkappa dz[t],\ w[t_0] = x_0 \tag{3.6}$$

generated by the pair $(U_1^e, V^{**}) \in \{U\}_1 \times \{V\}_1$. The equality

$$w_\omega[t] = w_* + \int_{\tau_i}^{t} f(\tau, w_\omega[\tau], u_\omega[\tau], \quad \vartheta_\omega[\tau])\, d\tau + \int_{\tau_i}^{t} \varkappa dz_\omega[\tau] \tag{3.7}$$

$$u_\omega[\tau] = U_1^e(\tau, w_\omega[\tau]), \quad v_\omega[\tau] = V^{**}(\tau, w_\omega[\tau], u_\omega[\tau]) = V^*(\tau, x^*[\tau], w_\omega[\tau], u_\omega[\tau])$$ is fulfilled with

probability one for this solution. Thus, we have defined the probabilistic process $(x_\omega[t], w_\omega[t])$ $(t_0 \leqslant t \leqslant \tau_{i+1}, \omega \in \Omega_*)$ for a fixed realization $(x_{\omega_*}[t], w_{\omega_*}[t])$ $(t_0 \leqslant t \leqslant \tau_i)$ of process $(x_\omega[t], w_\omega[t])$ $(t_0 \leqslant t \leqslant \tau_i, \omega \in \Omega_{\tau_i})$. Using the results [12] and taking into account that U_1^e, V^*, U_* and V_1° are Borel functions, we can correctly define the random process $(x_\omega[t], w_\omega[t])(t_0 \leqslant t \leqslant \tau_{i+1}, \omega \in \Omega_{\tau_{i+1}})$. To complete the recurrent definition of the probabilistic process it remains to note that on the initial interval $[t_0, \tau_1]$ the probabilistic process $(x_\omega[t], w_\omega[t])$ is defined by relations (3.5) and (3.7) wherein $\tau_i = t_0$, and $w_* = x_* = x_0$. The following statement holds.

L e m m a 3.1. The estimate

$$M[\| x[\vartheta] - w[\vartheta] \|^2] \leqslant [1 + (\vartheta - t_0)] \alpha(\varkappa, \delta) \exp[2\lambda(\vartheta - t_0)] \tag{3.8}$$

$$(\delta = \sup_i(\tau_{i+1} - \tau_i) \quad (i = 0, 1, \ldots, N-1); \ \alpha(\varkappa, \delta) \to 0$$

for $\varkappa \to 0$, $\delta \to 0$)
is valid for the random process $(x_\omega[t], w_\omega[t])$ $(t_0 \leqslant t \leqslant \vartheta, \omega \in \Omega_\vartheta)$ constructed. Here λ is the Lipschitz constant for function f with respect to variable x.

To prove the lemma we can first estimate the conditional mathematical expectation $M[\| x[\tau_{i+1}] - w[\tau_{i+1}] \|^2]$ for the fixed realization $(x_*[t], w_*[t])(t_0 \leqslant t \leqslant \tau_i)$. We obtain the estimate

$$M[\| x[\tau_{i+1}] - w[\tau_{i+1}] \|^2] \leqslant \| x_*[\tau_i] - w_*[\tau_i] \|^2 (1 + 2\lambda(\tau_{i+1} - \tau_i)) + \alpha(\varkappa, \delta)(\tau_{i+1} - \tau_i)$$

In the proof of this inequality we use the fact that the functions U_* of (3.4) and V^* of (3.3), which prescribe the choice of control u_* in system (3.1) and of control v^* in system (3.6), are determined from the conditions of mutual tracking of components $x[t]$, $w[t]$ of the random process $(x[t], w[t])$. Next we derive inequality (3.8) by using the formula for conditional mathematical expectations ([4]). The following statement is valid as well.

L e m m a 3.2. The estimate

$$M\sigma(w[\vartheta]) \leqslant \gamma_{1,\varkappa}(t_0, x_0) \tag{3.9}$$

is valid for the random process $(x_\omega[t], w_\omega[t])$ $(t_0 \leqslant t \leqslant \vartheta, \omega \in \Omega_\vartheta)$.

In the proof of this inequality (as in the proof of Theorem 2.1) we use the fact that control $u_\omega[t]$ $(t_0 \leqslant t \leqslant \vartheta)$ (see (3.7)) is formed by the strategy U_1^e

determined by condition (2.2). Since function σ satisfies a Lipschitz condition,

$$\sigma(x_\omega[\vartheta]) \leqslant \sigma(w_\omega[\vartheta]) + L \| x_\omega[\vartheta] - w_\omega[\vartheta] \|$$
$$M\sigma(x[\vartheta]) \leqslant M\sigma(w[\vartheta]) + LM \| x[\vartheta] - w[\vartheta] \|$$

From (3.8) it follows that $M \| x[\vartheta] - w[\vartheta] \| \leqslant \beta_*(\varkappa, \delta)$, where $\beta_*(\varkappa, \delta) \to 0$ as $\varkappa \to 0$ and $\delta \to 0$. Therefore, according to (3.9), we have

$$M\sigma(x[\vartheta]) \leqslant \gamma_{1,\varkappa}(t_0, x_0) + L\beta_*(\varkappa, \delta) = \gamma_{1,\varkappa}(t_0, x_0) + \beta(\varkappa, \delta) \ (\beta(\varkappa, \delta) \to 0 \ \text{for} \ \varkappa \to 0 \ \text{and} \ \delta \to 0) \tag{3.10}$$

According to the results in [1] the optimal counter-strategy V_1^0 ensures the fulfilment of the inequality

$$\sigma(x_\Delta[\vartheta; t_0, x_0, u[\cdot], V_1^0]) \geqslant c_1(t_0, x_0) - \varepsilon(\delta)$$
$$(\varepsilon(\delta) \to 0 \ \text{for} \ \delta \to 0)$$

under any choice of measurable realization $u[t] \in P \ (t_0 \leqslant t \leqslant \vartheta)$ (here $x_\Delta[t; t_0, x_0, u[\cdot], V_1^\circ]$ is an Euler polygonal line corresponding to counter-strategy V_1° and to control $u[\cdot]$, see [1]). Therefore, for the probabilistic process being analyzed, where the control $v[t]$ in system (3.1) is formed by the strategy V_1° we have the valid inequality

$$M\sigma(x[\vartheta]) \geqslant c_1 - \varepsilon(\delta)$$

Hence from (3.10) we obtain

$$\gamma_{1,\varkappa}(t_0, x_0) \geqslant c_1 - \beta(\varkappa, \delta) - \varepsilon(\delta) = c_1 - \alpha(\varkappa, \delta) \to c_1$$
$$\text{as} \ \varkappa \to 0, \ \delta \to 0$$

We have thus proved inequality (3.2) for $k = 1$. The inequality

$$\overline{\lim_{\varkappa \to 0}} \gamma_{k\varkappa}(t_0, x_0) \leqslant c_k(t_0, x_0) \tag{3.11}$$

with $k = 1$ can be proved similarly. To do this we should determine Borel functions U^* and V_* analogously to the functions introduced in the proof of inequality (3.2), replacing v^* by v_* and u_* by u^* in conditions (3.3) and (3.4).

To determine the probabilistic process $(x_\omega[t], w_\omega[t]) \ (t_0 \leqslant t \leqslant \vartheta)$ we now assume that the constant controls $u[\tau_i]$ and $v[\tau_i]$ in (3.5) are determined by the equalities $u[\tau_i] = U_1^\circ(\tau_i, x_*)$ and $v[\tau_i] = V_*(\tau_i, x_*, w_*, u[\tau_i])$, where U_1° is the optimal strategy in the position game defined in the class $\{U\}_1 \times \{V\}_1$ for system (3.1). In (3.7) we set $u_\omega[\tau] = U^*(\tau, x_\omega[\tau], w_\omega[\tau])$, and $v_\omega[\tau] = V_1^e(\tau, w_\omega[\tau], u_\omega[\tau])$, where $V_1^e \in \{V\}_1$ is the counter-strategy chosen from condition (2.5). Lemma 3.1 remains valid for the probabilistic process defined in this manner, while instead of inequality (3.9) the opposite inequality fulfilled. Further, the inequality $M\sigma(x[\vartheta]) \leqslant c_1 + \varepsilon(\delta)$, where $\varepsilon(\delta) \to 0$ as $\delta \to 0$, is fulfilled by the choice of strategy U_1°. Inequality (3.11) with $k = 1$ follows from these relations. From (3.2) and (3.11) we have that the limit of the quantity $\gamma_{1,\varkappa}(t_0, x_0)$ as $\varkappa \to 0$ exists and equals the value $c_1(t_0, x_0)$.

Let us describe briefly the construction of the probabilistic process $(x_\omega[t], \ w_\omega[t])$ used in the proof of inequality (3.2) with $k = 2$. We consider Borel functions $\bar{U}_*$ and $\bar{V}^*$ whose values satisfy, respectively, the conditions

$$\max_{\nu\in\bar{Q}}\int_P\int_Q (x-w)' f(t, x, u, v)\,\mu_*(du)\,\nu(dv) = F_2(t, x, x-w) \tag{3.12}$$

$$\min_{\mu\in\bar{P}}\int_P\int_Q (x-w)' f(t, x, u, v)\,\mu(du)\,\nu^*(dv) = F_2(t, x, x-w) \tag{3.13}$$

$$(\bar{U}_*(t, x, w) = \mu_* \in \bar{P}, \quad \bar{V}^*(t, x, w) = \nu^* \in \bar{Q})$$
$$(U_*: [t_0, \vartheta]\times R^n\times R^n \to \bar{P}, \quad \bar{V}^*: [t_0, \vartheta]\times R^n\times R^n \to \bar{Q})$$

The motion $x^*[t](\tau_i \leqslant t \leqslant \tau_{i+1})$ of system (3.1) is determined by the equality

$$x^*[t] = x_* + \int_{\tau_i}^{t}\int_P\int_Q f(\tau, x^*[\tau], u, v)\,\mu_\tau(du)\,\nu_\tau(dv)\,d\tau \tag{3.14}$$
$$\mu_\tau = \mu_{\tau_i} = \bar{U}_*(\tau_i, x_*, w_*), \quad \nu_\tau = \nu_{\tau_i} = V_2^\circ(\tau_i, x_*)$$

where V_2° is the optimal strategy in the position game defined in the class $\{U\}_2 \times \{V\}_2$ for system (3.1). Let function U_2^e be determined from condition (2.3) and function $\bar{V}^{**}$ by the equality $\bar{V}^{**}(t, w) = \bar{V}^*(t, x, w)$, where $\bar{V}^*$ is the function chosen from condition (3.13); then the pair $(U_2^e, \bar{V}^{**}) \in \{U\}_2 \times \{V\}_2$. Let us consider the motion of stochastic system (3.6), generated by the pair $(U_2^e, \bar{V}^*)$. The equality

$$w_\omega[t] = w_* + \int_{\tau_i}^{t}\int_P\int_Q f(\tau, w_\omega[\tau], u, v)\,\mu_{\omega,\tau}(du)\,\nu_{\omega,\tau}(dv)d\tau + \int_{\tau_i}^{t}\varkappa dz_\omega[\tau] \tag{3.15}$$
$$\mu_{\omega,\tau} = U_2^e(\tau, w_\omega[\tau]), \quad \nu_{\omega,\tau} = \bar{V}^*(\tau, x_\omega[\tau], w_\omega[\tau])$$

is fulfilled with probability one for the weak solution $w_\omega[t] = w_\omega[t; \tau_i, w_*, U_2^e, \bar{V}^*]$. From this point on the construction of the probabilistic process $(x_\omega[\tau], w_\omega[\tau])$ $(t_0 \leqslant t \leqslant \vartheta, \quad \omega \in \Omega_\vartheta)$ is similar to the construction for the case $k = 1$.

In the proof of inequality (3.11) with $k = 2$ we use Borel functions $\bar{U}^*$ and $\bar{V}_*$ chosen from conditions analogous to (3.12) and (3.13), as well as the functions U_2° and V_2^e, where U_2° is the optimal strategy in the position game defined in class $\{U\}_2\times\{V\}_2$ for system (3.1) and $V_2^e \in \{V\}_2$ is the mixed strategy chosen from condition (2.6).

In the proof of inequality (3.2) with k = 3 we use the functions defined in the proof of inequality (3.11), with the corresponding change of letters u, U, P to v, V, Q the replacement of condition (2.5) by (2.4) and of index $k = 1$ by index $k = 3$. In the proof of inequality (3.11) with $k = 3$ we use the functions defined in the proof of inequality (3.2), with the same interchange of letters and the replacement of condition (2.2) by (2.7) and of index $k = 1$ by index $k = 3$. Thus we have the following valid statement.

T h e o r e m 3.1. The limit of the value $\gamma_{k,\varkappa}(t_0, x_0)$ of the stochastic differential game defined in the strategy class $\{U\}_k \times \{V\}_k$ exists and equals the value $c_k(t_0, x_0)$ of the position differential game defined for the same type of strategies.

4. Let us show that the stochastic process (3.6) can be taken as a guide for the ordinary controlled system (3.1) when solving position game problems on the minimax and maximin of the payoff $\sigma(x[\vartheta])$. The control procedures constructed below ensure results arbitrarily close to the best with probability arbitrarily close to one.

To be specific let us consider the problem facing the first player and construct the control procedures ensuring the first player the fulfilment of the inequality

$$P[\sigma(x[\vartheta]) \leqslant c_k(t_0, x_0) + \varepsilon] \geqslant 1 - \varepsilon \quad (k = 1, 2, 3) \tag{4.1}$$

for a preselected $\varepsilon > 0$. Here, in the cases $k = 1$, we examine the situation when the first player knows the current position $(t, x[t])$, being his external information on the motion of controlled system (3.1). The second player can realize counter-strategies $V \in \{V\}_1$. In the case of $k = 2$ the first player's external information also is the knowledge of position $(t, x[t])$, while the opponent chooses the mixed strategies $V \in \{V\}_2$. In case $k = 3$ the first player knows the position $(t, x[t])$ and the opponent's current control $v[t]$, while the second player can realized a pure strategy $V \in \{V\}_3$. In the case $k = 1$ and $k = 3$ the first player realizes a pure control $u[t] \in P$ and in the case $k = 2$, a mixed control $\mu_t \in \bar{P}$.

F i r s t p l a y e r ' s c o n t r o l p r o c e d u r e i n c a s e $k = 1$. We admit the possibility of solving Eq. (2.1) for any $\varkappa > 0$. The estimates on $\alpha(\varkappa)$ from Sect. 3, bounding from above the distance between the numbers c_k and $\gamma_{k,\varkappa}$, can be given explicit expression. We assume here that for any $\varkappa > 0$ we know the quantity $\alpha(\varkappa)$ for which

$$|c_1 - \gamma_{1,\varkappa}| \leqslant \alpha(\varkappa); \quad \alpha(\varkappa) \to 0, \quad \varkappa \to 0 \tag{4.2}$$

From the specified $\varepsilon > 0$ we determine the number $\varkappa_1 > 0$ for which the inequality $\alpha(\varkappa_1) \leqslant {}^1/_8\,\varepsilon^2$ is fulfilled. We solve the parabolic Eq. (2.1) with $\varkappa = \varkappa_1$ and we determine $\gamma_{1,\varkappa_1}(t_0, x_0)$. The value c_1 of the position differential game with payoff $\sigma(x[\vartheta])$ satisfies the inequality

$$\gamma_{1,\varkappa_1}(t_0, x_0) - {}^1/_8\varepsilon^2 \leqslant c_1(t_0, x_0) \leqslant \gamma_{1,\varkappa_1}(t_0, x_0) + {}^1/_8\varepsilon^2 \tag{4.3}$$

Let us change function σ in the following manner. Let $D = \{x : \sigma(x) \leqslant \gamma_{1,\varkappa_1} + {}^1/_8\varepsilon^2\}$, we set

$$\sigma^*(x) = \begin{cases} \gamma_{1,\varkappa_1}(t_0, x_0) + {}^1/_8\varepsilon^2, & x \in D \\ \sigma_1(x)(\sigma(x) - {}^1/_2\varepsilon \leqslant \sigma_1(x) \leqslant \sigma(x)), & x \overline{\in} D \end{cases} \tag{4.4}$$

where function σ_1 is chosen so that function σ^* belongs to the class indicated in Sect. 1 and the condition within parentheses above is fulfilled. We see that with the changed payoff function σ^* the value (denoted c_1^*) of the position differential game in the strategy class $\{U\}_1 \times \{V\}_1$ equals

$$c_1^* = \gamma_{1,\varkappa_1}(t_0, x_0) + {}^1/_8\, \varepsilon^2 \tag{4.5}$$

Let $\gamma^*_{\mathrm{I},\varkappa}$ be a solution of Eq. (2.1) with the boundary condition $\gamma^*_{\mathrm{I},\varkappa}(\vartheta, x) = \sigma^*(x)$. We note that here in the definition of function σ^* the parameter $\varkappa_1$ is assumed to be fixed, while the choice of parameter $\varkappa$ in function $\gamma^*_{\mathrm{I},\varkappa}$ will be indicated below. Analogously to (4.2) we have the condition

$$|\, c_1^* - \gamma^*_{\mathrm{I},\varkappa} \,| \leqslant \alpha(\varkappa)$$

Hence, allowing for equality (4.5) and inequality (4.3), we obtain the estimate

$$\gamma^*_{\mathrm{I},\varkappa}(t_0, x_0) \leqslant c_1 + {}^1/_4\varepsilon^2 + \alpha(\varkappa) \tag{4.6}$$

Note that the estimate (3.10) holds for the functions σ^* of (4.4) and $\gamma^*_{\mathrm{I},\varkappa}$ being examined here. We now choose parameters $\delta_* > 0$ and $\varkappa_* > 0$ so as to fulfil the inequalities

$$\beta(\varkappa_*, \delta_*) \leqslant {}^1/_8\varepsilon^2, \qquad \alpha(\varkappa_*) \leqslant {}^1/_8\varepsilon^2 \tag{4.7}$$

For this choice of functions σ^* and $\gamma^*_{\mathrm{I},\varkappa_*}$ and of parameters δ_* and $\varkappa_*$ the proposed first player's control procedure is the following. Suppose that the first player selected a partitioning Δ_1 of interval $[t_0, \vartheta]$ by points $t_0 = \tau_0^{(1)} < \tau_1^{(1)} < \ldots < \tau_M^{(1)} = \vartheta$, satisfying the condition $\tau_{m+1}^{(1)} - \tau_m^{(1)} \leqslant \delta_*$. Let Δ_2 be a partitioning of interval $[t_0, \vartheta]$ by points $t_0 = \tau_0^{(2)} < \tau_1^{(2)} < \ldots < \tau_N^{(2)} = \vartheta$, selected by the second player.

The motions $x_\omega[t] (t_0 \leqslant t \leqslant \vartheta)$ are generated by controls $u[t]$ and $v[t]$ chosen by the first and second players at the instants $\tau_m^{(1)}$ $(m = 0, 1, \ldots, M-1)$ and $\tau_j^{(2)}$ $(j = 0, 1, \ldots, N-1)$, respectively, on the half-intervals $[\tau_m^{(1)}, \tau_{m+1}^{(1)})$ and $[\tau_j^{(2)}, \tau_{j+1}^{(2)})$ by the formulas

$$u[t] = u[\tau_m^{(1)}] = U_*(\tau_m^{(1)}, x_\omega[\tau_m^{(1)}], w_\omega[\tau_m^{(1)}]) \tag{4.8}$$

$$v[t] = v[\tau_j^{(2)}] = V(\tau_j^{(2)}, x_\omega[\tau_j^{(2)}], u[t]) \tag{4.9}$$

Here the function U_* is defined by condition (3.4) and $V \in \{V\}_1$ is some counter-strategy selected by the second player.

As in Sect. 3 the probabilistic process $(x_\omega[t], w_\omega[t])$ $(t_0 \leqslant t \leqslant \vartheta, \omega \in \Omega_\vartheta)$ is constructed recurrently, also by first defining certain auxiliary processes. We examine a partitioning Δ of interval $[t_0, \vartheta]$ by points $t_0 = \tau_0 < \tau_1 < \ldots < \tau_H = \vartheta$ including the points of partitionings Δ_1 and Δ_2. We remark that here the partitionings Δ_1 and Δ_2 do not coincide; therefore, the functions $(x_\omega[t], w_\omega[t], u_\omega[t], v_\omega[t])$ $(t_0 \leqslant t \leqslant \tau_i)$ are the realizations of the auxiliary probabilistic processes, and not, as in Sect. 3, simply the motions $(x_\omega[t], w_\omega[t]) (t_0 \leqslant t \leqslant \tau_i)$.

We determine motion $x^*[t]$ from equality (3.5) in which control $u[\tau]$ is determined by (4.8) for $\tau_m^{(1)} \leqslant \tau_i < \tau_{m+1}^{(1)}$ and control $v[\tau]$ by (4.9) for $\tau_j^{(2)} \leqslant \tau_i < \tau_{j+1}^{(2)}$. In the auxiliary probabilistic process the equality $x_\omega[t] = x^*[t]$ is fulfilled with probability one for the motions $x_\omega[t], w_\omega[t]$ for $\tau_i \leqslant t \leqslant \tau_{i+1}$, The random motions $w_\omega[t]$ $(\tau_i \leqslant t \leqslant \tau_{i+1})$ in the auxiliary process are determined from equality (3.7) in which the parameter $\varkappa = \varkappa_*$ is chosen from condition (4.7) and strategy $U_1{}^e$ is determined by condition (2.2) in which function γ_1 should be replaced by function $\gamma^*_{1,\varkappa_*}$. It can be verified that the probabilistic process $(x_\omega[t], w_\omega[t])(t_0 \leqslant t \leqslant \vartheta, \ \omega \in \Omega_\vartheta)$ can indeed be defined when the motions $x[t]$ and $w[t]$ are defined as indicated. Furthermore, the estimates in Sect. 3 remain valid for this probability process. In particular, the inequality

$$M\sigma^*(x[\vartheta]) \leqslant \gamma^*_{1,\varkappa_*}(t_0, \ x_0) + \beta(\varkappa_*, \ \delta_*) \qquad (4.10)$$

holds. From (4.10), (4.6) and (4.7) we obtain

$$M\sigma^*(x[\vartheta]) \leqslant c_1(t_0, \ x_0) + {}^1/_2\varepsilon^2 \qquad (4.11)$$

Since $\sigma^*(x[\vartheta]) \geqslant c_1$ for all x, from (4.11) follows

$$P[\sigma^*(x[\vartheta]) \leqslant c_1 + {}^1/_2\varepsilon] \geqslant 1 - \varepsilon$$

whence, by virtue of (4.4), we have the inequality (4.1) with k = 1.

First player's control procedure in case $k = 2$. Inequality (4.1) with $k = 2$ can be proved analogously. The relations (4.2)-(4.7) derived for the case $k = 1$ remain valid with index $k = 1$ replaced by index $k = 2$. In the construction of motion $x_\omega[t]$ $(t_0 \leqslant t \leqslant \vartheta)$ we determine the probability measures $\mu_t(du)$ and $\nu_t(dv)$ selected by the first and second players at instants $\tau_m{}^{(1)}$ and $\tau_j{}^{(2)}$, respectively, on the half-intervals $[\tau_m^{(1)}, \tau_{m+1}^{(1)})$ and $[\tau_j^{(2)}, \tau_{j+1}^{(2)})$ by the formulas

$$\mu_t(du) = \mu_{\tau_m^{(1)}}(du) = \bar{U}_*(\tau_m^{(1)}, x_\omega[\tau_m^{(1)}], w_\omega[\tau_m^{(1)}]) \qquad (4.12)$$

$$(m = 0, 1, \ldots, M-1)$$

$$\nu_t(dv) = \nu_{\tau_j^{(2)}}(dv) = V(\tau_j^{(2)}, x_\omega[\tau_j^{(2)}]) \quad (j = 0, 1, \ldots, N-1) \qquad (4.13)$$

Here function $\bar{U}_*$ is determined by condition (3.2) and $V \in \{V\}_2$ is a mixed strategy chosen by the second player.

The motion $x^*[t]$ $(\tau_i \leqslant t \leqslant \tau_{i+1})$ in the auxiliary probabilistic process is determined from equality (3.14) in which the function $\mu_\tau(du)$ is determined by (4.12) for $\tau_m^{(1)} \leqslant \tau_i < \tau_{m+1}^{(1)}$ and the function $\nu_\tau(dv)$ by (4.13) for $\tau_j^{(2)} \leqslant \tau_i < \tau_{j+1}^{(2)}$. The random motions $w_\omega[t]$ $(\tau_i \leqslant t \leqslant \tau_{i+1})$ are determined from equality (3.15) in which the parameter $\varkappa = \varkappa_*$ and the strategy $U_2{}^e$ is determined by condition (2.3) in which function γ_2 should be replaced by function $\gamma^*_{2,\varkappa_*}$. The first player's control procedure in the case $k = 3$ is shown below.

We pass on to the problem facing the second player and we construct the control procedures ensuring the second player the fulfilment of the inequality

$$P[\sigma(x[\vartheta]) \geqslant c_k - \varepsilon] \geqslant 1 - \varepsilon \quad (k = 1,2,3) \qquad (4.14)$$

for a preselected $\varepsilon > 0$. We change function σ as follows:

$$\sigma^{**}(x) = \begin{cases} \gamma_{k,\varkappa_1}(t_0, x_0) - {}^1/_8\varepsilon^2, & x \overline{\in} D \\ \sigma_2(x)(\sigma(x) \leqslant \sigma_2(x) \leqslant \sigma(x) + {}^1/_2\varepsilon), & x \in D \end{cases}$$

where function σ_2 is chosen so that function σ^{**} belongs to the class indicated in Sect. 1 and the condition within parentheses above is fulfilled.

Second player's control procedure in the case $k=1$. Analogously to (4.6) we can obtain the estimate

$$\gamma_{1,\varkappa}^{**}(t_0, x_0) \geqslant c_1(t_0, x_0) - {}^1/_4\varepsilon^2 - \alpha(\varkappa)$$

where $\gamma_{1,\varkappa}^{**}$ is a solution of Eq. (2.1) with the boundary condition $\gamma_{1,\varkappa}^{**}(\vartheta, x) = \sigma^{**}(x)$. We assume that the second player has selected a partitioning Δ_2 of the interval $[t_0, \vartheta]$, satisfying the condition $\tau_{j+1}^{(2)} - \tau_j^{(2)} \leqslant \delta_*$, $j = 0, 1, \ldots, N-1$, and that the first player has selected partitioning Δ_1 of interval $[t_0, \vartheta]$ by points $t_0 = \tau_0^{(1)} < \tau_1^{(1)} < \ldots < \tau_M^{(1)} = \vartheta$.

The motions $x_\omega[t](t_0 \leqslant t \leqslant \vartheta)$ are generated by controls $u[t]$ and $v[t]$, chosen by the first and second players at instants $\tau_m^{(1)}$ $(m = 0, 1, \ldots, M-1)$ and $\tau_j^{(2)}$ $(j = 0, 1, \ldots, N-1)$ respectively, on the half-intervals $[\tau_m^{(1)}, \tau_{m+1}^{(1)})$ and $[\tau_j^{(2)}, \tau_{j+1}^{(2)})$ by the formulas

$$u[t] = u[\tau_m^{(1)}] = U(\tau_m^{(1)}, x_\omega[\tau_m^{(1)}]) \tag{4.15}$$

$$v[t] = v[\tau_j^{(2)}] = V_*(\tau_j^{(2)}, x_\omega[\tau_j^{(2)}], u[t])$$

Function V_* is determined by the condition in Sect. 3 and $U \in \{U\}_1$ is a pure strategy chosen by the first player.

The motion $x^*[t]$ $(\tau_i \leqslant t \leqslant \tau_{i+1})$ in the auxiliary probabilistic process is determined by equality (3.5) in which the controls $u[\tau]$ and $v[\tau]$ are chosen in accord with (4.15) for $\tau_m^{(1)} \leqslant \tau_i < \tau_{m+1}^{(1)}$ and $\tau_j^{(2)} \leqslant \tau_i < \tau_{j+1}^{(2)}$. The random motions $w_\omega[t](\tau_i \leqslant t \leqslant \tau_{i+1})$ are determined from equality (3.7) in which $u_\omega[\tau] = U^*(\tau, x_\omega[\tau], w_\omega[\tau])$ and $v_\omega[\tau] = V_1^e(\tau, w_\omega[\tau], u_\omega[\tau])$. Here function U^* has been defined in Sect. 3 and strategy V_1^e is chosen from condition (2.5) in which function γ_1 should be replaced by function $\gamma_{1,\varkappa_*}^{**}$. Further, once again it can be verified that the probabilistic process $(x_\omega[t], w_\omega[t])$ $(t_0 \leqslant t \leqslant \vartheta, \omega \in \Omega_\vartheta)$ can indeed be defined when the motions $x[t]$ and $w[t]$ are defined as indicated and the bounds in Sect. 3 remain valid for it. In particular, the inequality

$$M\sigma^{**}(x[\vartheta]) \geqslant \gamma_{1,\varkappa_*}^{**}(t_0, x_0) - \beta(\varkappa_*, \delta_*)$$

holds. Hence, by arguing analogously as in the proof of inequality (4.1), we obtain inequality (4.14) with $k=1$.

S e c o n d p l a y e r ' s c o n t r o l p r o c e d u r e i n t h e c a s e $k=2$. To prove inequality (4.14) the second player's control procedure is constructed analogously to the first player's control procedure in the proof of inequality (4.1) in the case $k=2$.

S e c o n d p l a y e r ' s c o n t r o l p r o c e d u r e i n t h e c a s e $k=3$. When constructing the second player's control procedure we use the functions defined in the first player's control procedure when proving inequality (4.1) for $k=1$, with the corresponding change of letters u, U, P to v, V, Q, the replacement of index $k=1$ by index $k=3$, of condition (2.2) by (2.7) and of function γ_3 by $\gamma^{**}_{3,\varkappa_*}$.

F i r s t p l a y e r ' s c o n t r o l p r o c e d u r e i n t h e c a s e $k=3$. To prove inequality (4.1), in the first player's procedure we use the functions determined in the proof of inequality (4.14) for the case $k=1$, with the corresponding change of letters u, U, P to v, V, Q, the replacement of index $k=1$ by index $k=3$, of condition (2.5) by (2.4) and of function γ_3 by $\gamma^{*}_{3,\varkappa_*}$. The following statement is valid.

T h e o r e m 4.1. The initial position (t_0, x_0) is specified. For any preselected $\varepsilon > 0$ we can find $\varkappa > 0$ and $\delta > 0$ for which the control procedures constructed guarantee the first and second players the fulfilment of the inequalities

$$P[\sigma(x[\vartheta]) \leqslant c_k + \varepsilon] \geqslant 1 - \varepsilon, \quad P[\sigma(x[\vartheta]) \geqslant c_k - \varepsilon] \geqslant 1 - \varepsilon$$

respectively, where $c_k = c_k(t_0, x_0)$ is the value of the position differential game defined in the strategy class $\{U\}_k \times \{V\}_k$.

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